

The grand canonical catastrophe as an instance of condensation of fluctuations

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PACS 05.30.Jp – Boson systems

PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion

PACS 05.30.Ch – Quantum ensemble theory

Abstract – The so-called grand canonical catastrophe of the density fluctuations in the ideal Bose gas is shown to be a particular instance of the much more general phenomenon of condensation of fluctuations, taking place in a large system, in or out of equilibrium, when a single degree of freedom makes a macroscopic contribution to the fluctuations of an extensive quantity. The pathological character of the “catastrophe” is demystified by emphasizing the connection between experimental conditions and statistical ensembles, as demonstrated by the recent realization of photon condensation under grand canonical conditions.

Introduction. – The grand canonical behavior of the density fluctuations of an ideal Bose gas (IBG) is known to contradict the intuitive notion that fluctuations ought to be suppressed by lowering the temperature. This is commonly referred to as the *grand canonical catastrophe* [1–3], alluding to a failure of the grand canonical ensemble (GCE) for bosons in the condensed phase. Furthermore, the successful realization of Bose-Einstein condensation (BEC) of ultracold atoms, which requires the canonical or microcanonical framework, has contributed to consolidate the notion of naivety and inadequacy of the GCE. Yet, the GCE results are admittedly exact. Hence, the alleged flaws of the GCE do not arise from internal inconsistencies, but from the supposedly unphysical nature of the conditions required for the GCE to apply. However, with the recent experimental realization of BEC in a gas of photons [4], complemented shortly after by the investigation of the condensate fluctuations [5], the outlook has changed, because condensation in this case has been achieved under grand canonical conditions. This is an important development with implications on the issue of statistical ensembles, since the grand canonical catastrophe has been proven to be a real physical effect.

In this paper we approach the fluctuation problem in the framework of large deviation theory and we show that the observation of the grand canonical catastrophe is of inter-

est in a context much wider than the physics of bosons, as an instance of *condensation of fluctuations* [6–8]. Condensation of fluctuations is a phenomenon related to but distinct from the usual condensation which appears after taking a thermodynamic *average*. The latter, to be referred to as condensation on average, belongs to the realm of typical behavior, while the former normally is a rare event. These different instances of condensation may appear jointly or disjointly. Of particular interest is the case of non interacting systems in which the distinction between the two becomes extreme and, therefore, most clear since without interaction there cannot be condensation on average and yet fluctuations may condense [7]. In this respect, the IBG framework offers a rich and flexible scenario allowing to cover both cases: of condensation of fluctuations with and without condensation on average. This paper aims to show that when the two types of condensation occur simultaneously, the grand canonical catastrophe, rather than being a pathology, is the observable manifestation of condensation of fluctuations.

In order to explain what condensation of fluctuations is about, it is convenient to lay down the basic probabilistic structure: Consider a generic system with microscopic states ω in a macrostate described by the statistical ensemble $P(\omega|J, V)$, where J stands for a set of control parameters and V for the size of the system. The setting is very general: ω could be a single event in sample space as well as a trajectory, in which case V would involve the

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trajectory's time length. Here, since we have in mind an equilibrium application, V will be taken as the system's volume. Referring to this ensemble as the prior, let $\mathcal{M}(\omega)$ be an extensive observable which scales like V . Now, the probability that $\mathcal{M}(\omega)$ takes the value M is given by

$$P(M|J, V) = \sum_{\omega} P(\omega|J, V) \delta(\mathcal{M} - M). \quad (1)$$

Though straightforward, this turns out to be a statement rich of consequences since a relation is established between the observation of a fluctuation in the prior and the imposition of a constraint on the same system. In fact, using terminology from statistical mechanics, $P(M|J, V)$ plays also the role of the partition function in the constrained ensemble

$$P_c(\omega|M, J, V) = \frac{1}{P(M|J, V)} P(\omega|J, V) \delta(\mathcal{M} - M), \quad (2)$$

obtained by conditioning the prior to the event $\mathcal{M}(\omega) = M$. If $\mathcal{M}(\omega)$ obeys a large deviation principle, the above duality between fluctuations and constraints is reflected into the twofold role of the rate function

$$I(m|J) = - \lim_{V \rightarrow \infty} \frac{1}{V} \ln P(M|J, V), \quad m = M/V \quad (3)$$

as the entity controlling fluctuations in the prior and as the free energy density in the constrained ensemble [9].

The formal framework is completed by the important observation that the task of computing the rate function can be simplified resorting to a soft version of the constraint (for details see [9] or [7]), namely by applying an exponential bias on the prior in place of the hard δ function constraint. This yields the biased ensemble

$$P_b(\omega|s, J, V) = \frac{1}{K(s, J, V)} P(\omega|J, V) e^{s\mathcal{M}(\omega)} \quad (4)$$

where $K(s, J, V) = \sum_{\omega} P(\omega|J, V) e^{s\mathcal{M}(\omega)}$ is the biased partition function. Continuing with the language of statistical mechanics, the constrained ensemble (2) describes the system enclosed by walls isolating with respect to \mathcal{M} , while the biased ensemble describes the system in contact with an \mathcal{M} -reservoir. Then, the biased free energy $Y(s, J) = \lim_{V \rightarrow \infty} \frac{1}{V} \ln K(s, J, V)$ and $I(m|J)$ are related by the Legendre transformation

$$I(m|J) = s^* m - Y(s^*, J) \quad (5)$$

where s^* is the biasing field such that

$$\langle \mathcal{M} \rangle_{s^*} = m \quad (6)$$

and where $\langle \cdot \rangle_s$ stands for the average in the biased ensemble. Summarizing, through $I(m|J)$ a link is established between fluctuations in the prior and typical behavior either in the constrained $P_c(\omega|M, J, V)$ or in the biased ensemble $P_b(\omega|s^*, J, V)$, which are equivalent in the sense of ensemble theory.

The implication of this duality is that rare fluctuations, difficult to observe, can be made typical by the implementation of constraints [10]. Conversely, if constraints are difficult or even impossible to realize in practice, in principle constrained system could be studied through the fluctuations of unconstrained ones. The key point, for what follows, is that fluctuations in the unconstrained system explore the phase diagram of the constrained one.

The probabilistic nature of the above structure makes it of wide applicability. Among the many applications, as anticipated above, of particular interest is the apparently trivial case of an uncorrelated prior. In fact, the imposition of a constraint induces correlations which, in turn, may cause phase transitions to occur in the constrained system. Perhaps, the best known example of this type is the ferromagnetic transition due to the spherical constraint imposed on the Gaussian model [11]. Then, as a consequence of duality, the transition singularity appears also in the behavior of the prior's fluctuations while the prior's typical behavior, due to the absence of correlations, is smooth and featureless by construction. In this case there will be condensation of fluctuations without condensation on average. This phenomenon has been studied in different contexts such as information theory [12], finance [13] and statistical mechanics [6–8], encompassing both equilibrium and out of equilibrium behavior.

Grand canonical vs mean canonical ensemble. –

The rest of the paper is devoted to the study of the implications of the general structure expounded above in the context of the IBG in equilibrium. Focus will be on the boson number fluctuations. In order to make the presentation as simple as possible, a uniform system in a d -dimensional box of volume V with periodic boundary conditions will be considered. The microstates ω are the sets of occupation numbers $\omega = \{n_{\vec{p}}\}$ of the single particle momentum eigenstates. The energy function is separable $\mathcal{H}(\omega) = \sum_{\vec{p}} n_{\vec{p}} \epsilon_p$ and the single particle dispersion relation is of the form $\epsilon_p = a p^\alpha$, where a is a proportionality constant. For instance, for photons $a = c$ velocity of light and $\alpha = 1$, while for particles with mass m , $a = 1/(2m)$ and $\alpha = 2$.

The statistical ensemble is determined by the system's preparation protocol. Thus, in the GCE which applies when the system is put in contact with a thermal and a particle reservoir, the probability of a microstate is given by

$$P_{gc}(\omega|\beta, \mu, V) = \frac{1}{Z_{gc}(\beta, \mu, V)} e^{-\beta[\mathcal{H}(\omega) - \mu \mathcal{N}(\omega)]}, \quad (7)$$

where $\mathcal{N}(\omega) = \sum_{\vec{p}} n_{\vec{p}}$ is the number function. For large enough V , the equation of state takes the form [14]

$$\rho = \frac{1}{V} \frac{z}{1-z} + \lambda^{-d} g_\nu(z), \quad (8)$$

where ρ is the average density, $z = e^{\beta\mu}$ is the fugacity, λ

is the thermal wavelength [15],

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty dx \frac{x^{\nu-1} z}{e^x - z} \quad (9)$$

is the Bose function with $\nu = d/\alpha$ and Γ is the Euler gamma function. In the above formula the Bose function collects the contributions to the density from the excited states, while the V -dependent term is due to the average occupation of the ground state. In the $\nu > 1$ case, to which we will restrict from now on, $g_\nu(1) = \zeta(\nu)$, where ζ is the Riemann zeta function. Hence, in the $V \rightarrow \infty$ limit the isotherms become superiorly bounded by the critical value

$$\rho_C(\beta) = \lambda^{-d} g_\nu(1). \quad (10)$$

This means that, by taking the thermodynamic limit *and* adopting the GCE protocol, it is not possible to prepare the system with an average density above ρ_C . In other words, BEC does not take place in the GCE, as it is well known from the example of black body radiation, corresponding to the GCE with $z = 1$. Occurrence of BEC, as condensation on average, requires number conservation which can be implemented either rigidly or softly. In the first case, the number of particles is fixed and the ensemble statistics is canonical

$$P_c(\omega|\beta, N, V) = \frac{1}{Z_c(\beta, N, V)} e^{-\beta \mathcal{H}(\omega)} \delta_{N, N}. \quad (11)$$

In the second case the number of particles is fixed on average [14] by inverting the density-fugacity relation. This amounts to introduce the new ensemble defined by

$$P_{mc}(\omega|\beta, \rho, V) = P_{gc}(\omega|\beta, z^*, V), \quad (12)$$

where the function $z^*(\beta, \rho, V)$ is the unique root of Eq. (8) given by (see Appendix 1)

$$\ln z^*(\beta, \rho, V) = \begin{cases} \ln g_\nu^{-1}(\lambda^d \rho), & \text{for } \rho < \rho_C, \\ -AV^{-1/\nu}, & \text{for } \rho = \rho_C, \\ -1/[V(\rho - \rho_C)], & \text{for } \rho > \rho_C, \end{cases} \quad (13)$$

with $A = [-\lambda^d \Gamma(\nu) \sin(\pi\nu)/\pi]^{1/\nu}$. In the following, GCE will denote only the ensemble controlled by (β, z, V) , while the ensemble controlled by (β, ρ, V) will be referred to as the *mean canonical* ensemble (MCE), adopting terminology from the spherical model literature [16]. Though formally similar, these ensembles become non equivalent [17] in the thermodynamic limit: Dividing the density axis into the normal phase below ρ_C and the condensed phase above ρ_C , the GCE and MCE overlap in the first one but not in the second, which is accessible to the average density only in the MCE.

It should also be noted that the origin of BEC in the MCE is in the mean-field treatment of the correlations generated by number conservation, with $\mu^* = \ln z^*$ playing a role akin to that of the internal Weiss field in the mean-field theory of ferromagnetism. Once this is understood, the distinction between MCE and GCE becomes

quite clear, since in the first ensemble the absence of interaction is formal, while the second one is genuinely non-interacting. This is the same distinction existing between the mean spherical model and the Gaussian model of magnetic systems [11, 16]. The similarities between the spherical model and the IBG have been pointed out by Kac and Thompson in the paper of Ref. [16]. On the experimental side, the control on the boson number is one of the key elements in the realization of BEC in the laboratory. In experiments with ultracold atoms condensation is achieved in (micro)canonical conditions with control parameters (β, N, V) by enclosing a definite number of atoms in optical traps [3]. Instead, in the work of Klaers et al. [4], photons condensation has been obtained operating with the MCE protocol, that is by keeping fixed the average density ρ [18].

Therefore, it is of interest to study the density fluctuations in the two ensembles, since on the basis of the considerations made in the Introduction, in the GCE (or in the MCE normal phase) condensation of fluctuations without condensation on average is expected, while in the MCE condensed phase the two types of condensation are expected to appear simultaneously.

Density fluctuations. – As a consequence of the equivalence of the two ensembles in the normal phase, the study of fluctuations can be unified and carried out in the MCE framework for both of them.

Using Eq. (12), the probability to find the value N of \mathcal{N} for a given ρ reads

$$P_{mc}(N|\beta, \rho, V) = \sum_\omega P_{gc}(\omega|\beta, z^*, V) \delta_{N, N}. \quad (14)$$

We are interested in checking whether \mathcal{N} obeys a large deviation principle and, if so, to find the rate function. This requires to extract the V -independent part from $\mathcal{I}_{mc} = -\frac{1}{V} \ln P_{mc}$, evaluated for large V . Carrying out the algebra expounded in Appendix 2 and denoting the fluctuating density by $x = N/V$, for $x \leq \rho_C$ one obtains

$$\begin{aligned} \mathcal{I}_{mc}(x|\beta, \rho, V) &= x \ln \frac{z^*(x)}{z^*(\rho)} + \frac{1}{V} \ln z^*(x) \\ &+ \frac{1}{V} \ln \frac{Z_{gc}(\beta, z^*(\rho), V)}{Z_{gc}(\beta, z^*(x), V)}, \end{aligned} \quad (15)$$

with the function z^* defined by Eq. (13). Instead, for $x > \rho_C$ one finds the linear behavior

$$\mathcal{I}_{mc}(x|\beta, \rho, V) = (\rho_C - x) \ln z^*(\rho) + \mathcal{I}_{mc}(\rho_C|\beta, \rho, V). \quad (16)$$

Here, two comments are in order. The first one is about duality, which appears in a simplified form in Eq. (15) since \mathcal{I}_{mc} is related to the partition function in the *same* ensemble $Z_{mc}(\beta, x, V) = Z_{gc}(\beta, z^*(x), V)$. The reason is that by taking the MCE as prior the bias needed to render typical the fluctuation x amounts just to shift the fugacity from $z^*(\rho)$ to $z^*(x)$, without changing the form of the ensemble. Hence, with respect to number fluctuations, the

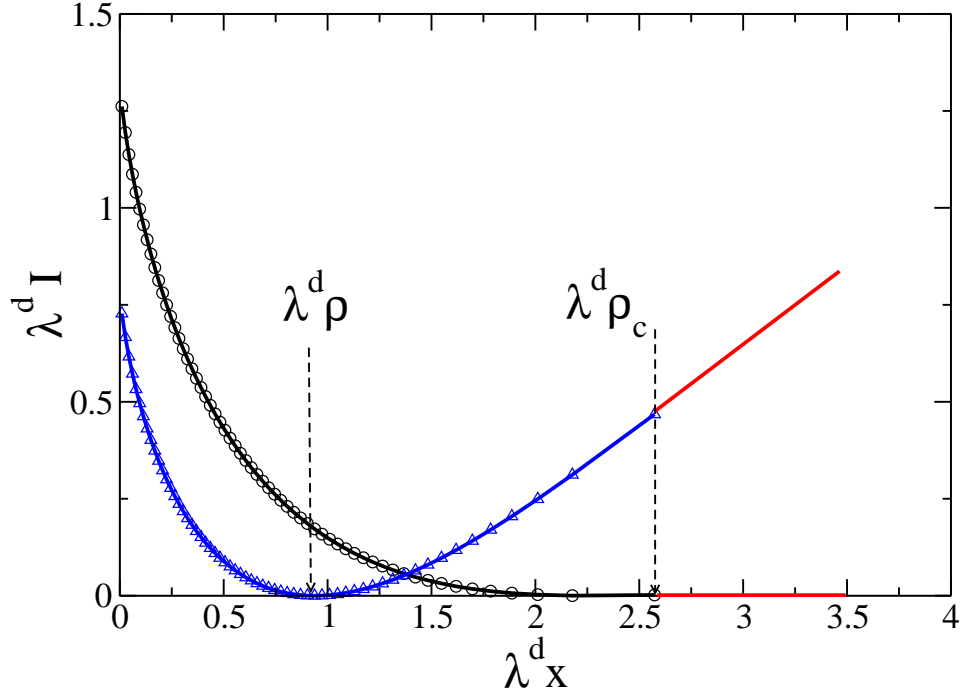


Fig. 1: Rescaled large deviation function vs rescaled fluctuating density in the MCE: normal phase (blue triangles) with $-\ln z^* = 0.4$ and $\nu = 3/2$, condensed phase (black circles) with $\rho > \rho_C$ (Color on line). It is a straightforward computation to show that the temperature dependence is rescaled away by the λ^d factor.

MCE is *self-dual*. The second one is that the linear branch of Eq. (16) corresponds to condensation of fluctuations. In fact, exponentiating and comparing with the probability of the ground state occupation number

$$P_0(n_0|\beta, z, V) = (1 - z)e^{n_0 \ln z}, \quad (17)$$

there follows that for $x > \rho_C$ and up to normalization one can write

$$P_{mc}(N|\beta, \rho, V) = P_0(N - N_C|\beta, z^*(\rho), V)P_{mc}(N_C|\beta, \rho, V), \quad (18)$$

which shows that macroscopic fluctuations above threshold can occur *only* through the zero momentum occupation number, while the contribution of the excited states is locked to $N_C = V\rho_C$.

The next step is to analyze separately the above result when ρ is in the normal or in the condensed phase. We shall see that the $V \rightarrow \infty$ limit yields different behaviors in the two cases.

Normal phase. As follows from Eq. (13), in the normal phase $z^*(\rho) < 1$. Furthermore, $z^*(\rho)$ is independent of V . Thus, letting $V \rightarrow \infty$, there exists the rate function

$$\begin{aligned} I_{mc}(x|\beta, \rho) &= \lim_{V \rightarrow \infty} \mathcal{I}_{mc}(\rho|\beta, \rho, V) \\ &= \begin{cases} x \ln \frac{z^*(x)}{z^*(\rho)} + \beta[f_{mc}(\beta, x) - f_{mc}(\beta, \rho)], & x \leq \rho_C \\ (\rho_C - x) \ln z^*(\rho) + I_{mc}(\rho_C|\beta, z^*(\rho)), & x > \rho_C, \end{cases} \end{aligned} \quad (19)$$

where the MCE free energy density is given by [14] $f_{mc}(\beta, \rho) = -\beta^{-1} \lambda^{-d} g_{\nu+1}(z^*(\rho))$. As the plot displayed

in Fig. 1 (curve with triangles) shows, this is a convex function with an isolated minimum at the typical value ρ , where it vanishes, which is strictly convex for $x \leq \rho_C$ and linear for $x > \rho_C$.

Hence, in the normal phase of the MCE (or in the GCE) when x exceeds the critical threshold fluctuations do condense in absence of condensation on average. Though interesting, this fluctuation phenomenon is doomed to remain an hardly observable event, since it is exponentially suppressed for large V . In fact, due to the existence of the isolated minimum at ρ

$$\lim_{V \rightarrow \infty} P_{mc}(x|\beta, \rho, V) = \delta(x - \rho). \quad (20)$$

As we shall see, this situation is drastically changed in the condensed phase where condensation of fluctuations coexists with condensation on average.

Condensed phase. When ρ is in the condensed phase, from Eq. (13) follows that $z^*(\rho)$ becomes V dependent with $\lim_{V \rightarrow \infty} z^*(\rho) = 1$. Hence, from Eqs. (15) and (16) follows

$$\begin{aligned} I_{mc}(x|\beta, \rho) &= \\ &\begin{cases} x \ln z^*(x) + \beta[f_{mc}(\beta, x) - f_{mc}(\beta, \rho_C)], & x \leq \rho_C \\ 0, & x > \rho_C \end{cases} \end{aligned} \quad (21)$$

which shows that the large deviation function vanishes identically for fluctuations above threshold [8] (see Fig. 1). This means that when both ρ and x are in the condensed phase, fluctuations behave sub-exponentially with respect

to V and that in order to get the probability of fluctuations V dependent terms must be retained. Putting together the first line of Eq. (21) and lowest order terms from Eq. (16) one has

$$\mathcal{I}_{\text{mc}}(x|\beta, \rho, V) = \begin{cases} I_{\text{mc}}(x|\beta, \rho), & x \leq \rho_C \\ \frac{1}{V} \left[\left(\frac{x - \rho_C}{\rho - \rho_C} \right) + \ln(V(\rho - \rho_C)) \right], & x > \rho_C. \end{cases} \quad (22)$$

Hence, due to the $1/V$ factor in the second line of the above equation, the probability of a fluctuation for $x > \rho_C$ becomes V independent and, in the $V \rightarrow \infty$ limit, instead of the δ function of Eq. (20) now one obtains the distribution

$$P_{\text{mc}}(x|\beta, \rho) = \begin{cases} 0, & \text{for } x \leq \rho_C, \\ \frac{\exp\{-\frac{x - \rho_C}{\rho - \rho_C}\}}{(\rho - \rho_C)}, & \text{for } x > \rho_C. \end{cases} \quad (23)$$

which can be recognized as the Kac function of Ref. [1] and which is also the statement of condensation of fluctuations, as it can be checked by comparison with Eq. (17). This is the main result in the paper.

With such a broad distribution the distinction between typical and rare events gets blurred and the width of fluctuations becomes macroscopic, since it goes like the density of the condensate

$$\sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \rho - \rho_C. \quad (24)$$

Such a regime of strong fluctuations persists and is even enhanced by lowering the temperature, since ρ_C vanishes as $\beta \rightarrow \infty$. Therefore, these features, which constitute the grand canonical catastrophe, are nothing but the manifestation of condensation of fluctuations which, in the condensed phase, is as typical as condensation on average.

Conclusions. – The possibility to observe such a strong fluctuation regime depends only on the realization of the physical conditions specific of the MCE. Even though the experiment by Klaers et al. [4] is modeled by bosons trapped in an harmonic potential, yet the conceptual structure expounded above still holds suggesting that the large fluctuations of the condensate reported in Ref. [5] are due to condensation of fluctuations.

Summarizing, density fluctuations of the ideal Bose gas in thermal equilibrium have been analyzed in the normal phase (which is equivalent to GCE) and in the condensed phase of the MCE. In the first case condensation of fluctuations takes place as a rare event and in absence of condensation on average. In the second case the two types of condensation occur simultaneously, both as typical events and producing the phenomenology of the grand canonical catastrophe, which, therefore, should not be regarded as a pathology, but just as an observable feature of the MCE. The natural question is whether the conditions responsible of these macroscopic fluctuations could be realized in atomic systems. Typically, atoms confined in optical traps are in microcanonical conditions and, therefore, the overall density does not fluctuate. However, one could

think to have such a large system that a subsystem could be regarded in a GCE with energy and particle reservoirs due to the rest of the system. Then, if the chemical potential of the subsystem is controlled through the density ρ_{ext} of the outer system, it was shown by Ziff et al. [1] that $\lim_{V \rightarrow \infty} P(x|\beta, \rho_{\text{ext}}, V) = \delta(x - \rho_{\text{ext}})$ for ρ_{ext} in the normal as well as in the condensed phase. Hence, there would be no grand canonical catastrophe. However, it has been pointed out by Schmitt et al. [5] that this prediction is not in conflict with their findings, because it pertains to experimental conditions in which the subsystem is in diffusive contact with the environment, which is not the case for the photon experiment. The point is that the experiment of Schmitt et al. takes place in genuine MCE conditions, meaning by this that the control parameter is the average density ρ of the system *itself*, as opposed to ρ_{ext} in the conditions envisaged by Ziff et al. Though the distinction may seem rather subtle, yet it is enough to produce different statistical ensembles as demonstrated by the drastically different fluctuation regimes in the condensed phase.

Appendix 1. – Derivation of Eq. (13): If $\rho < \rho_C$ the first term in the right hand side of Eq. (8) can be neglected, trivially yielding the V independent solution in the first line of Eq. (13). Instead, if $\rho \geq \rho_C$ one has $\epsilon = -\ln z \ll 1$. Therefore, using the expansions $z \simeq 1 - \epsilon$ and [19]

$$g_\nu(z) = g_\nu(1) + \Gamma(1 - \nu)\epsilon^{\nu-1} \quad (25)$$

Eq. (8) can be rewritten as

$$\rho - \rho_C = \frac{1}{V\epsilon} + \frac{\Gamma(1 - \nu)}{\lambda^d} \epsilon^{\nu-1}, \quad (26)$$

from which, it is straightforward to obtain the second and third line of Eq. (13), under the assumption $\nu < 2$, i.e. $d < 2\alpha$, with $\Gamma(1 - \nu) = \pi/[\Gamma(\nu)\sin(\pi\nu)]$. Notice that the condition on ν is fulfilled for particles with mass and with $d < 4$. In the experimental conditions of Refs. [4, 5] photons do behave as particles with an effective positive mass.

Appendix 2. – Using the integral representation of the Kronecker δ

$$\delta_{N, \mathcal{N}} = \oint_{\mathcal{C}} \frac{dz'}{2\pi i} z'^{\mathcal{N}-N-1}, \quad (27)$$

the probability (14) takes the form

$$P_{\text{mc}}(N|\beta, \rho, V) = e^{-V\Phi(x|\beta, z^*(\rho), V)} \oint_{\mathcal{C}} \frac{dz'}{2\pi i} \frac{1}{z'} e^{V\Phi(x|\beta, z', V)}, \quad (28)$$

where \mathcal{C} is a complex contour enclosing the origin and

$$\Phi(x|\beta, z, V) = -x \ln z + \frac{1}{V} Z_{\text{gc}}(\beta, z, V), \quad (29)$$

with

$$Z_{\text{gc}}(\beta, z, V) = -\frac{1}{V} \ln(1 - z) + \lambda^{-d} g_{\nu+1}(z). \quad (30)$$

Hence,

$$\begin{aligned}\mathcal{I}_{\text{mc}}(x|\beta, \rho, V) &= -\frac{1}{V} \ln P_{\text{mc}}(N|\beta, \rho, V) \\ &= \Phi(x|\beta, z^*(\rho), V) + K(x|\beta, V),\end{aligned}\quad (31)$$

where

$$K(x|\beta, V) = -\frac{1}{V} \ln \oint_C \frac{dz'}{2\pi i} \frac{1}{z'} e^{V\Phi(x|\beta, z', V)}. \quad (32)$$

The saddle point of the integrand function is determined by $\frac{\partial}{\partial z'} \Phi(x|\beta, z', V) = 0$, which yields the equation

$$x = \frac{1}{V} \langle \mathcal{N} \rangle_{z'}, \quad (33)$$

whose solution $z^*(\beta, x, V)$, therefore, depends on x according to Eq. (13). Hence, $z^*(x)$ is independent of V for $x < \rho_C$ and V -dependent for $x \geq \rho_C$. In the first case, from Eq. (32) follows straightforwardly

$$K(x|\beta, V) = -\Phi(x|\beta, z^*(x), V) + \frac{1}{V} \ln z^*(x). \quad (34)$$

In the second one, since $z^*(x) = e^{-1/[V(x-\rho_C)]} \simeq 1$ with a very weak dependence on x , the integral is dominated by the region where the Bose function is well approximated by [19]

$$g_{\nu+1}(z') \simeq g_{\nu+1}(1) + g_{\nu}(1) \ln z'. \quad (35)$$

As a check, one can verify that the saddle point equation obtained with the above substitution coincides, in the region of interest, with Eq. (33). Then, using the above form of $g_{\nu+1}(z')$ the integral can be carried out exactly, yielding

$$K(x|\beta, V) = -\lambda^{-d} g_{\nu+1}(1), \quad (36)$$

which is independent of x . Inserting these results into Eq. (31), one finds

$$\begin{aligned}\mathcal{I}_{\text{mc}}(x|\beta, \rho, V) &= \\ &\Phi(x|\beta, z^*(\rho), V) - \Phi(x|\beta, z^*(x), V) + \frac{1}{V} \ln z^*(x),\end{aligned}\quad (37)$$

for $x < \rho_C$ and

$$\mathcal{I}_{\text{mc}}(x|\beta, \rho, V) = \Phi(x|\beta, z^*(\rho), V) - \lambda^{-d} g_{\nu+1}(1), \quad (38)$$

for $x \geq \rho_C$, which coincide with Eqs. (15) and (16).

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